Solutions for problem set 1

1. From

$$R(N) = \frac{\delta\gamma}{\gamma - 1}\left(\frac{N}{\delta}\right)^{(\gamma-1)/\gamma}$$

it follows that

(a)

$$\begin{aligned} R(0) &= \frac{\delta\gamma}{\gamma-1}\left(\frac{0}{\delta}\right)^{(\gamma-1)/\gamma} = 0, \\ R'(N) &= \frac{\delta\gamma}{\gamma-1}\frac{\gamma-1}{\gamma}\left(\frac{N}{\delta}\right)^{(\gamma-1)/\gamma-1}\frac{1}{\delta} \\ &= \left(\frac{N}{\delta}\right)^{-1/\gamma} > 0 \\ R''(N) &= -\frac{1}{\gamma}\left(\frac{N}{\delta}\right)^{-1/\gamma-1}\frac{1}{\delta} < 0. \end{aligned}$$

The fact that $R''(N) < 0$ for $N > 0$ means that marginal revenues decrease if N increases. As N grows very large, marginal revenues converge to 0. It becomes harder and harder to substantially increase revenues by only adjusting the employment level N and keeping all other inputs fixed.

(b) The labour demand curve is the locus of points maximizing the profit function. The first-order condition for profit maximization is $R'(N) = w$ (marginal revenue equals marginal cost). The solution is $N(w) = \delta w^{-\gamma}$.

(c) The wage-elasticity of labour demand is given by

$$N'(w)\frac{w}{N(w)} = -\gamma < 0.$$

(d) As can be seen from this last expression, any increase in δ (shifting the labour demand function upwards) leaves the wage elasticity unchanged.

2. As the utility function is given by $u(w) = w^{\alpha}$,

(a) $u'(w) = \alpha w^{\alpha-1} > 0$ for all $\alpha > 0$: workers appreciate higher wages.
From $u''(w) = \alpha(\alpha-1) w^{\alpha-2}$, and $\alpha > 0$, it can be seen that, for $w > 0$,

$$u''(w) \gtreqless 0 \Leftrightarrow \alpha \gtreqless 1.$$

Thus, workers are risk loving for $\alpha > 1$, risk neutral for $\alpha = 1$, and risk averse for $0 < \alpha < 1$ (see you microeconomics text book for an explanation and graphical illustration of expected utility theory and risk attitudes). The assumption of risk-aversion or risk-neutrality for the remainder of the problem set is equivalent to the assumption that $0 < \alpha \leq 1$. In particular, this guarantees that $\gamma > 1 > \alpha > 0$, which we will use throughout.

(b) From the definition of Arrow-Pratt measure of relative risk aversion,

$$-\frac{w}{u'(w)} u''(w) = -\frac{w}{\alpha w^{\alpha-1}} \alpha(\alpha-1) w^{\alpha-2} = 1 - \alpha.$$

The lower is α, the more risk averse is the worker. This is consistent with (a).

(c) Individual labour supply is 40 hours a week, independently of the level of wage (in this exercise). However, individual labour supply to the *sector* will depend on how the sectoral wage w compares to the wage w_0 outside the sector. If $w > w_0$ then the worker supplies the 40 hours to the sector. For $w < w_0$ the worker prefers to work outside the sector. For $w = w_0$ she/he is indifferent between working in the sector or elsewhere.

(d) In competitive equilibrium, sectoral wages equal wages elsewhere in the economy: $w = w_0$ and $N = N(w_0)$.

3. Now, a trade union controls sectoral labour supply.

(a) The objective function of an utilitarian union is to maximize the sum of its M members' utilities $V(w, N) = Nu(w) + (M - N) u(w_0)$. The utility function can also be written as

$$V(w, N) = N(u(w) - u(w_0)) + Mu(w_0),$$

which shows that, for given union size M and outside wage w_0, maximizing the union utility function V is equivalent to maximizing the rent $N(u(w) - u(w_0))$. If all union workers are treated the same in the job allocation process, each union member faces a probability N/M of being allocated a union job (for $N \leq M$). So, the expected level of utility of a union member before union jobs are distributed (*ex ante*) is

$$\frac{N}{M} u(w) + \left(1 - \frac{N}{M}\right) u(w_0) = \frac{V(w, N)}{M}.$$

So, $V(w,N)$ can be interpreted as the sum of the ex ante expected utilities of the M union members. Alternatively, after jobs have been distributed among union members (*ex post*), N of them will have utility level $u(w)$, whereas $M-N$ of them, will end up with $u(w_0)$. So, as we already knew, $V(w,N)$ is also the sum of the ex post utilities of the union members.

(b) Without exerting its market power, the union can guarantee the market wage w_0 to its members. This corresponds to a minimal union utility level of $Mu(w_0)$. The corresponding indifference curve is the horizontal line $w = w^0$: union members are indifferent between working in the union sector and working elsewhere because the wages are the same. For higher utility levels $\bar{V} > Mu(w_o)$, the iso-utility curves satisfy (for $N \leq M$)

$$\begin{aligned} & Nu(w) + (M-N)u(w_0) = \overline{V} \\ \Leftrightarrow\quad & N(u(w) - u(w_0)) = \overline{V} - Mu(w_0) \\ \Leftrightarrow\quad & w = u^{-1}\left(\frac{\overline{V} - Mu(w_0)}{N} + u(w_0)\right) = \left(\frac{\overline{V} - Mu(w_0)}{N} + u(w_0)\right)^{1/\alpha}. \end{aligned}$$

In a graph (with, as always, w on the vertical axis and N on the horizontal axis), they have the usual downward sloping convex shape that we have seen in class.[1] Higher fixed utility values $\bar{V}$ correspond to higher indifference curves. Note that we have assumed in class that the indifference curves are flat for $N > M$: as long as every member of the trade union has a job, the trade union is only interested in increasing the wage level.

(c) Additional employment at wage w_0 is guaranteed for union workers, because they can always work in the competitive sector. Thus, the trade union will never agree upon a wage that is lower than w_0.

(d) The monopoly union maximizes its utility given the firm's labour demand $N(w)$. The maximal level of utility is therefore

$$\begin{aligned} & \max_w \{Nu(w) + (M-N)u(w_0) \quad \text{such that} \quad N = N(w)\} \\ = \quad & \max_w \{N(w)u(w) + (M - N(w))u(w_0)\}, \end{aligned}$$

for which an interior solution $(w_u, N(w_u))$ is achieved when the *first-order condition*

$$\begin{aligned} & N'(w_u)u(w_u) + N(w_u)u'(w_u) - N'(w_u)u(w_0) = 0 \\ \Leftrightarrow\quad & -\frac{N(w_u)}{N'(w_u)} = \frac{u(w_u) - u(w_0)}{u'(w_u)} \\ \Leftrightarrow\quad & -w_u\frac{N'(w_u)}{N(w_u)} = w_u\frac{u'(w_u)}{u(w_u) - u(w_0)}, \end{aligned}$$

[1] This can be formally checked by deriving the first and second derivatives of the indifference curves.

holds. This is McDonald and Solow's equation (2). Note that the left hand side is (minus the) wage-elasticity of labour demand. Substituting our results of exercises 1 and 2, we obtain

$$\begin{aligned}
&\gamma = \alpha \frac{w_u^\alpha}{w_u^\alpha - w_0^\alpha} \\
\Leftrightarrow \quad & \gamma \left(w_u^\alpha - w_0^\alpha \right) = \alpha w_u^\alpha \\
\Leftrightarrow \quad & w_u = \left(\frac{\gamma}{\gamma - \alpha} \right)^{\frac{1}{\alpha}} w_0 > w_0. \qquad (1)
\end{aligned}$$

(e) In the competitive case, an *iso-elastic* shift of the sectoral labour demand curve leads to an increase in employment. Wages are not affected because *sectoral* labour supply is perfectly elastic at the competitive wage w_0 (see 2(c)). As wages do not respond to demand shift, we can expect a strong impact on employment. In the monopoly-union case, we find similar effects: equation (1) shows that the monopoly-union wage w_u is not affected by iso-elastic shifts in labour demand (that is, shifts that leave γ unchanged).

(f) This is a bit more involved than intended. It is easiest to rewrite (1) in logs first, which gives

$$\ln(w_u) = \frac{1}{\alpha} \ln \left(\frac{\gamma}{\gamma - \alpha} \right) + \ln(w_0).$$

The signs of $\partial w_u / \partial \alpha$ and $\partial \ln(w_u) / \partial \alpha$ are the same (because $\ln(w_u)$ is strictly increasing in w_u), so we can focus on the latter, which is

$$\frac{\partial \ln(w_u)}{\partial \alpha} = -\frac{1}{\alpha^2} \ln \left(\frac{\gamma}{\gamma - \alpha} \right) + \frac{1}{\alpha(\gamma - \alpha)} = \frac{1}{\alpha^2} \left(\frac{\alpha}{\gamma - \alpha} - \ln \left(\frac{\gamma}{\gamma - \alpha} \right) \right).$$

This derivative is positive, because $x > \ln(1 + x)$ for all $x \neq 0$ (set $x = \alpha / (\gamma - \alpha)$). If union members are more risk averse (i.e. if α is *lower*; see 2(b)), then they settle for a lower wage in return for a higher probability of union employment.

(g) An increase in w_0 will have a positive impact on w_u (show that $\partial w_u / \partial w_0 > 0$, using equation (1)). If working outside the union sector is more attractive, the union accepts a higher risk for its members of being displaced from the union sector and sets a higher union wage.

(h) The monopoly-union wage is unaffected by any change in δ, so the full wage-effect of this change is captured by $\partial w_u / \partial \gamma$. As in (f), it is convenient to consider $\partial \ln(w_u) / \partial \gamma$ instead, which gives

$$\frac{\partial \ln(w_u)}{\partial \gamma} = \frac{1}{\alpha} \left(\frac{1}{\gamma} - \frac{1}{\gamma - \alpha} \right) = -\frac{1}{\gamma(\gamma - \alpha)} < 0.$$

If labour demand is more elastic, the monopoly-union will set lower wages to avoid loss of employment in the union sector.

If we vary γ, the sectoral labour demand function changes, which may lead to a change in the competitive equilibrium outcome. In particular, as the competitive sectoral wage is fixed at the given economy-wide wage w_0, it may change competitive sectoral employment $N_0 = N(w_0)$. We want to change δ along with γ so that $N(w_0)$ is not affected.

Graphically, this boils down to a counterclockwise rotation of the labour demand curve around (N_0, w_0) (the rotation itself corresponds to a increase in the absolute elasticity; by rotating around the competitive equilibrium, this equilibrium is unchanged). Clearly, this implies a fall in labour demand at the old monopoly-union wage level ($> w_0$). However, we have just shown that, in response, the monopoly wage itself decreases, which corresponds to a movement down and to the right along the new labour demand curve. This offsets some, or perhaps all, of the decrease in labour demand. We need some calculus to find the net effect of these two effects. This not so straightforward and can be skipped (the result is that there is a net increase in labour demand).

Recall that $N(w_0) = \delta w_0^{-\gamma}$ or, in logs, $\ln(N_0) = \ln(\delta) - \gamma \ln(w_0)$. In total-differential form, relating small changes in all variables and parameters, this gives

$$d\ln(N_0) = \frac{1}{\delta} d\delta - \ln(w_0) d\gamma - \gamma d \ln(w_0).$$

Here we have used that $d\ln(\delta)/d\delta = 1/\delta$. If we impose that $dw_0 = dN_0 = 0$, i.e. that (w_0, N_0) does not change, this gives (in obvious notation I hope)

$$\frac{d\delta}{d\gamma}\rfloor_{dw_0 = dN_0 = 0} = \delta \ln(w_0).$$

In words, if we consider a change of $d\gamma$ of (minus) the wage-elasticity of labour demand γ then we have to change δ by $d\delta = \delta \ln(w_0) d\gamma$ to ensure that the competive equilibrium is unchanged ($dw_0 = dN_0 = 0$). The effect of such a combined change in δ and γ on employment follows from substituting $d\delta = \delta \ln(w_0) d\gamma$, $\ln(w_u)$ and $\partial \ln(w_u)/\partial\gamma$ in the total differential for $N_u = N(w_u)$,

$$d\ln(N_u) = \frac{1}{\delta} d\delta - \ln(w_u) d\gamma - \gamma d \ln(w_u).$$

This gives

$$\begin{aligned} \frac{d\ln(N_u)}{d\gamma}\rfloor_{dw_0 = dN_0 = 0} &= \ln(w_0) - \ln(w_u) - \gamma \frac{\partial \ln(w_u)}{\partial \gamma} \\ &= \frac{1}{\alpha}\left(\frac{\alpha}{\gamma - \alpha} - \ln\left(\frac{\gamma}{\gamma - \alpha}\right)\right), \end{aligned}$$

which is positive (see (f)).

4. On efficiency.

   (a) The iso-profit curves are concave and have a unique maximum (but do not forget that the profit is the same at all points on the iso-profit curve). For a given wage $\overline{w}$, the firm seeks to position itself on the *lowest* iso-profit line (which corresponds to the *highest* profit level) compatible with $\overline{w}$ (see the figure given in the class). This iso-profit curve is tangent to the horizontal line at $\bar{w}$ at the profit-maximizing point, which satisfies (see derivation in lecture) $R'(N) = \overline{w}$ (to the left of this point $R'(N) > \overline{w}$, to the right $R'(N) < \overline{w}$). Note that the firm simply balances marginal revenue R' and marginal cost $\bar{w}$ of labour. This gives the optimal quantity of labour $N(\overline{w})$ at the given wage $\overline{w}$. Repeating this exercise for various wages $\overline{w}$, one obtains the labour demand function $N(w)$ as the locus of maximum points of the iso-profit curves (i.e., the labour demand curve cuts to the maximum points of all iso-profit curves).

   (b) At the monopoly-union point, there exist wage-employment points at which both parties are better off. Hence, this situation is not Pareto optimal. It is possible to increase profits and union's utility level by decreasing the wage and increasing the level of employment.

   Graphically, this follows from the fact that the iso-utility curve through the point (N_u, w_u) is tangent to the (downward-sloping) demand curve by the first-order condition of the union's wage setting problem. As the iso-profit curves have slope 0 on the labour demand curve, this implies that the iso-utility and iso-profit curves through (N_u, w_u) intersect (you should draw a graph here). So, it is possible to move to a higher iso-utility and lower iso-profit curve that have points in common. Each of these points delivers higher utility to the union and higher profit to the firm.

   (c) First of all, the competitive point is Pareto efficient. Second, we have seen in 3(b) that it delivers the lowest utility level that is acceptable to the union. So, it is on an extreme of the contract curve.

   (d) See, Figure 3, equation (3) and the discussion on page 900 in Ian M. McDonald and Robert M. Solow, 1981, "Wage Bargaining and Employment", *American Economic Review*, 71, pp. 896-908. Alternatively, check the discussion in Booth.

5. "How rail privatisation gave unions more power". The article claims that "the fragmentation of the industry gave his negotiators opportunities to play off one train operator against another in a spiral of wage inflation". One way of squeezing this into our models is as in increase in the union's bargaining power relative to the firm(s). In, for example, the (general) right-to-manage model, this would imply

that we move closer to the monopoly-union outcome and away from the competitive wage after the privatisation.

Aslef, the train driver's union, should be particularly successful in achieving this. After all, they represent workers with very specific skills that are hard to substitute in the short-term by other workers or capital. So, the wage-elasticity of the demand for their members is probably low. This means that the employment effects of high wage demands are relatively small (compared to less-skilled and easier-to-substitute workers in the rail sector).